Engineering Notes

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Motivating Kane's Method for Obtaining Equations of Motion for Dynamic Systems

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OVER the years, Kane's method has become extremely popular among dynamicists as an efficient technique for generating equations of motion for complex dynamical systems, rivaling classical approaches such as Newton-Euler and Lagrangian. Many papers have been written comparing Kane's method against alternate techniques with respect to algebraic complexity, computational efficiency, and ease of digital implementation. Such issues will not concern us here. The purpose of this Note is to fill a pedagogical void we feel exists in the presentation of Kane's method in the open literature.[1-3]

Typically, one considers a system of particles and writes Newton's law for each individual particle. This is followed by taking the dot product with the "partial velocity" and summing over the particles of the system. Although the partial velocity is well defined, its introduction into Newton's law is somewhat artificial. A reader encountering this presentation for the first time is apt to dismiss it as a clever trick. Of course, it is well known that the key advantage in introducing the partial velocities is the subsequent elimination of nonworking constraint forces—a fact that becomes clear only later in the presentation. The same situation occurs if one derives Lagrange's equations starting from Newton's law. Here, we take the dot product with the virtual displacement of the particle and sum over all particles. Given the transformation between Cartesian coordinates and generalized coordinates, we can express the virtual displacement of a particle as a linear combination of variations of the independent generalized coordinates, the coefficients being the partial velocities. The connection between the two approaches is clear.

In a recent paper,[4] Scott presents a technique he calls the "projection method" to obtain equations of motion for mechanical systems. The beauty of the method lies in its intuitive appeal as a direct generalization of methods used in simple dynamics problems. To introduce the technique, consider the motion of a simple pendulum and a particle moving on a smooth surface under the influence of gravity. The most elementary and direct solution to the preceding problems consists of resolving forces and accelerations along the tangent and normal directions to the particle path. The tangential components give the equations of motion, whereas the normal components determine the forces of constraint. In many situations, only the information obtained from the tangential projection is of interest. In the language of vector spaces, we are projecting Newton's law onto a subspace. In the case of the pendulum, the subspace is one-dimensional, whereas in the case of the particle constrained to the surface the subspace is two-dimensional—it consists of the local tangent plane to the surface at the location of the particle. Scott generalizes this concept to many-particle systems characterized by an independent set of generalized coordinates.

We extend Scott's method to nonholonomic systems and show that Kane's equations (for both holonomic and nonholonomic systems) can be viewed as a projection of Newton's law upon a tangent plane to a hypersurface.

Projection Method—Holonomic Case

As a means of introduction, consider a single particle system characterized by two *independent* generalized coordinates q_1,q_2. Let $\boldsymbol{x} = (x_1,x_2,x_3)^T$ denote the position vector of the particle relative to an inertial Cartesian reference. Regarding $\boldsymbol{x}$ as a function of the generalized coordinates and time, we can express the particle velocity as

$$\frac{d\boldsymbol{x}}{dt} = J\dot{\boldsymbol{q}} + \frac{\partial \boldsymbol{x}}{\partial t}$$

where $\dot{\boldsymbol{q}} = (\dot{q}_1,\dot{q}_2)^T$ and

$$J = \begin{bmatrix} \dfrac{\partial x_1}{\partial q_1} & \dfrac{\partial x_1}{\partial q_2} \\ \dfrac{\partial x_2}{\partial q_1} & \dfrac{\partial x_2}{\partial q_2} \\ \dfrac{\partial x_3}{\partial q_1} & \dfrac{\partial x_3}{\partial q_2} \end{bmatrix}$$

The transformation between Cartesian coordinates and generalized coordinates can be interpreted geometrically as a surface with q_1 and q_2 as curvilinear coordinates (see Fig. 1). In this situation it is clear that the tangent plane is spanned by the vectors: $\partial\boldsymbol{x}/\partial q_1$, $\partial\boldsymbol{x}/\partial q_2$, i.e., the columns of the Jacobian matrix J.

We now consider the general situation of an N particle system characterized by n independent generalized coordinates $q_1, q_2, \ldots, q_n$. Let $\boldsymbol{x}$ be a $p \times 1$ column matrix containing the Cartesian coordinates of the particles relative to an inertial frame ($p \geqslant n$). A particle may have one, two, or three Cartesian coordinates; hence, $N \leqslant p \leqslant 3N$. The $p \times 1$ matrix $\boldsymbol{F}$ are the forces acting upon the particles, and the $p \times p$ diagonal matrix M are the particle masses. Newton's law for the system can thus be written as $\boldsymbol{F} = M\ddot{\boldsymbol{x}}$. The equations relating Cartesian coordinates to generalized coordinates are given by

$$\boldsymbol{x} = X(q_1,q_2,\ldots,q_n,t) \tag{1}$$

It follows that the vector of particle velocities is given by

$$\dot{\boldsymbol{x}} = J\dot{\boldsymbol{q}} + \boldsymbol{b}$$

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where

$$b = \frac{\partial X}{\partial t}$$

and

$$J = \left[\frac{\partial X}{\partial q_1} \; \frac{\partial X}{\partial q_2} \cdots \frac{\partial X}{\partial q_n}\right]$$

Newton's law thus takes the form

$$F = M(J\ddot{q} + \dot{J}\dot{q} + \dot{b}) \tag{2}$$

Using the preceding example as a guide, we define the tangent space to be the subspace of R^p spanned by the columns of J. Under suitable restrictions on the vector-valued function X of Eq. (1) that are satisfied in practice, it can be shown that the columns of J are linearly independent. Thus, the columns of J form a basis for the tangent space. Our next task is to find the orthogonal projection of both members of Eq. (2) onto the tangent space of the hypersurface defined by Eq. (1). In order to effect this projection, we make use of the following theorem.

Theorem: The orthogonal projection of $z \in R_p$ onto the tangent space is $\mathbf{0}$ if and only if $J^T z = \mathbf{0}$.

Proof: Let $\{j_1, j_2, \ldots, j_n\}$ be an orthonormal basis for the tangent space. If the orthogonal projection of z is $\mathbf{0}$, then

$$(z, j_1)j_1 + (z, j_2)j_2 + \cdots + (z, j_n)j_n = \mathbf{0}$$

where $(.,.)$ denotes the standard inner product on R^p. Since the j's are linearly independent, we conclude that $(z, j_i) = 0$, $(i = 1,2,\ldots,n)$. It follows that z is orthogonal to every vector in the tangent space. This fact can be expressed algebraically by $(z, Jw) = 0$, for any $w \in R^n$. Equivalently, $(J^T z, w) = 0$ for all w; thus, $J^T z = \mathbf{0}$. The converse can be shown to hold by a similar argument.

We can now write the final set of differential equations of motion projected onto the tangent space as

$$J^T M J\ddot{q} + J^T M \dot{J}\dot{q} + J^T M \dot{b} = J^T F \tag{3}$$

With regard to the invertibility of the coefficient matrix of the generalized accelerations, we can prove the following:

Theorem: $J^T M J$ is singular if and only if the columns of J are linearly dependent.

Proof: Assume that the columns of J are linearly dependent. There then exists an $\alpha \neq \mathbf{0}$ such that $J\alpha = \mathbf{0}$. Thus, $(J^T M J)\alpha = \mathbf{0}$. This represents a system of n linear homogeneous equations in n unknowns with a nontrivial solution. It follows that $J^T M J$ must be singular.

Now assume that $J^T M J$ is singular. It follows that the system of n linear homogeneous equations $J^T M J\alpha = \mathbf{0}$ has a nontrivial solution. Premultiplying by α^T we obtain $(J\alpha, MJ\alpha) = 0$. But M is positive definite; hence, $J\alpha = \mathbf{0}$, $(\alpha \neq \mathbf{0})$. This implies that the columns of J are linearly dependent.

It often proves convenient to partition the matrices appearing in Eq. (3) by the following scheme. Let $x^{(i)}$ denote the coordinates of the ith particle so that

$$x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(N)} \end{bmatrix}, \quad J = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{bmatrix}, \quad F = \begin{bmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(N)} \end{bmatrix}, \quad b = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N)} \end{bmatrix}$$

and

$$M = \mathrm{diag}(M_1, M_2, \ldots, M_N)$$

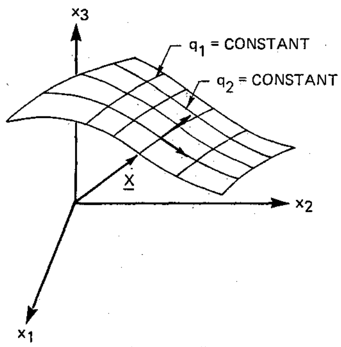

Fig. 1 Transformation Surface.

Equation (3) then assumes the form

$$\sum_{i=1}^{N} m_i J_i^T J_i \ddot{q} + \sum_{i=1}^{N} m_i J_i^T \dot{J}_i \dot{q} + \sum_{i=1}^{N} m_i J_i^T \dot{b}^{(i)} = \sum_{i=1}^{N} J_i^T f^{(i)} \tag{4}$$

where m_i is the mass of the ith particle.

We now show that Kane's equations are identical to the set derived previously by the projection method. Kane's equations may be written as [Ref. 1, Eq. (19)]

$$\sum_{i=1}^{N} V_r^{(i)} \cdot f^{(i)} + \sum_{i=1}^{N} V_r^{(i)} \cdot (-m_i a^{(i)}) = 0, \quad r = 1,2,\ldots,n \tag{5}$$

Here, $a^{(i)}$ is the inertial acceleration of the ith particle, and $V_r^{(i)}$ are the partial velocities. Resolving all vectors into the inertial frame and observing that

$$V_r^{(i)} = \frac{\partial x^{(i)}}{\partial q_r} = r\text{th column of } J_i$$

and

$$a^{(i)} = J_i \ddot{q} + \dot{J}_i \dot{q} + \dot{b}^{(i)}$$

we can transform Kane's equations into the form of Eq. (4).

Projection Method—Nonholonomic Case

We consider the same system as for the foregoing holonomic case, but the n generalized coordinates are no longer independent, being subject to the constraints

$$A\dot{q} = B \tag{6}$$

The matrices $A(m \times n)$ and $B(m \times 1)$ are functions of the generalized coordinates and time. These equations are independent, and we assume that the $\dot{q}$'s have been ordered so that the last m columns of A are linearly independent. The equations of constraint then admit to the partitioning

$$[A_1 \mid A_2] \begin{pmatrix} \dot{q}^{(1)} \\ \dot{q}^{(2)} \end{pmatrix} = B$$

where A_1 is $m \times (n - m)$, and A_2 is $m \times m$ and nonsingular. The first $(n - m)$ generalized velocities are in $\dot{q}^{(1)}$ (independent), and the last m are contained in $\dot{q}^{(2)}$ (dependent). It follows that

$$\dot{q}^{(2)} = -A_2^{-1} A_1 \dot{q}^{(1)} + A_2^{-1} B$$

or in indicial form

$$\dot{q}_{n-m+s} = \sum_{r=1}^{n-m} C_{sr} \dot{q}_r + d_s, \quad s = 1,2,\ldots,m$$

Differentiating Eq. (1) and eliminating the dependent generalized velocities we obtain

$$\frac{dx}{dt} = \hat{J}\dot{q}^{(1)} + \hat{b} \tag{7}$$

The rth column of the $p \times (n-m)$ matrix $\hat{J}$ is given by

$$\frac{\partial X}{\partial q_r} + \sum_{s=1}^{m} C_{sr} \frac{\partial X}{\partial q_{n-m+s}}, \quad r = 1,2,...,n-m$$

and

$$\hat{b} = \sum_{s=1}^{m} \frac{\partial X}{\partial q_{n-m+s}} d_s + \frac{\partial X}{\partial t}$$

We must proceed with caution in constructing a basis for the tangent space. Returning to the example given in the previous section, we saw that when q_1 and q_2 vary *independently*, the particles admissible path is a *surface*. If the variations in q_1 and q_2 are *dependent*, then the admissible path is a *curve* in the surface. The dimension of the tangent space is reduced from two to one. In the current situation, the constraints [Eq. (6)] reduce the number of independent variations of generalized coordinates from n to $n-m$.

In terms of variations of independent generalized coordinates, we have [cf. Eq. (7)]

$$\delta x = \hat{J} \delta q^{(1)}$$

where δx are the virtual displacements of the particles satisfying the instantaneous equations of constraint. The vector δx lies in the tangent space to the hypersurface. Again we see that the columns of the matrix appearing in the velocity transformation form a basis for the tangent space. Proceeding exactly as in the holonomic case, we can project the equations of motion onto the tangent space. Partitioning these equations by individual particles we obtain

$$\sum_{i=1}^{N} m_i \hat{J}_i^T \hat{J}_i \ddot{q}^{(1)} + \sum_{i=1}^{N} m_i \hat{J}_i^T \dot{\hat{J}}_i \dot{q}^{(1)} + \sum_{i=1}^{N} m_i \hat{J}_i^T \dot{\hat{b}}^{(i)} = \sum_{i=1}^{N} \hat{J}_i^T f^{(i)} \tag{8}$$

Kane's equations for a nonholonomic system are given by Eq. (5), with $r = 1,2,...,n-m$. The partial velocity $V_r^{(i)}$ is the coefficient of $\dot{q}_r$ in the expression for the velocity of particle "i" in terms of the independent generalized velocities. Resolving all vectors in Eq. (5) into the inertial frame and observing that $V_r^{(i)}$ is simply the rth column of $\hat{J}_i$, we see that Kane's equations become identical to Eq. (8)—obtained by the projection method.

Summary

A derivation of Kane's equations based on orthogonal projections is presented. Although less succinct than the traditional approach, it offers some insight into the physics embodied in the equations and appears as a natural generalization of methods used in elementary problems. It is hoped that those familiar with Kane's equations will find the current derivation illuminating and that instructors will adopt it into the classroom setting.

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References

[1]Kane, T. R., "Dynamics of Nonholonomic Systems," *Journal of Applied Mechanics*, Vol. 83, Dec. 1961, pp. 574–578.

[2]Likins, P. W., "Analytical Dynamics and Nonrigid Spacecraft Simulation," Jet Propulsion Lab., Pasadena, CA, TR-32-1593, July 1974.

[3]Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications*, McGraw-Hill, New York, 1985. Chap. 6.

[4]Scott, D., "Can a Projection Method of Obtaining Equations of Motion Compete with Lagrange's Equations?," *American Journal of Physics*, Vol. 56, May 1988, pp. 451–456.

Improved Time-Domain Stability Robustness Measures for Linear Regulators

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Introduction

THE robustness of multivariable control systems, i.e., their ability to maintain performance in the face of uncertainties and perturbations, has been extensively studied in the past. The recently published literature on robustness analysis of linear time-invariant systems can be viewed from three perspectives: 1) the frequency domain approach, 2) the time domain approach, and 3) the frequency domain approach, which uses a state space representation of the system.

Although many of the robustness criteria developed so far are in the frequency domain,[1-5] it is also useful to analyze stability robustness of multivariable control systems in the time domain, especially when a broader class of parameter perturbations have to be considered, which appears in the state equations describing the plant. The time domain approach[6-12] is primarily based on the Lyapunov theory and generally involves checking only a finite number of inequalities, often just one, while the frequency domain methodology requires that all criteria over the whole range of frequencies be satisfied.

Recently, new bounds on linear time-invariant perturbations that do not destabilize the system were given, based on the frequency domain approach that uses a state space representation of the system.[13-14] It was shown that these bounds are superior to time-domain stability robustness criteria[6,11,12] in the sense that they are less conservative and that they can be applied to a more general class of systems and perturbations.

In this Note, we present a new time domain stability criterion for linear state space models. A computationally effective algorithm is proposed, which leads to perturbation bounds that are superior to those based on the frequency domain approach[13,14] and the time-domain approach.[6,11,12]

Problem Formulation and Development

Consider a linear time-invariant model of a physical system with linear time-invariant perturbations

$$\dot{x}(t) = (A + E)x(t) \tag{1}$$

where x is the n-dimensional state vector, A is an $n \times n$ asymptotically stable matrix, and E is a perturbation matrix. Two types of perturbations have been considered: unstructured perturbations and structured perturbations.

In what follows, a brief discussion on some common approaches to the stability robustness analysis of state space models is given. We restrict our attention only to structured perturbations.

Theorem 1[12]: Assume that the elements of the perturbation matrix E, Eq. (1), are restricted so that

$$|E_{ij}| \leq e_{ij} \tag{2}$$

and let $e = \max_{i,j} e_{ij}$. Then the system (1) is stable if

$$e < \frac{1}{\sigma_{\max}[(|P|U)_s]} = \mu_y \tag{3}$$

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